Proof-of-principle reconstruction and geometric benchmarking of 3-qubit pure states on ibm_osaka

T. Humera,[*, 1] B. P. Govindaraja,[†, 2] M. Taj,[*, 3] Sudha,[†,¶, 4] and A. R. Usha Devi[*,¶, 5]
* *Department of Physics, Bangalore University, Bangalore- 560056, India*
† *Department of Physics, Kuvempu University, Shankaraghatta-577451, India*
¶ *Inspire Institute Inc., Alexandria, Virginia, 22303, USA.*

Abstract: *We report a proof-of-principle experimental demonstration of the reconstruction and geometric benchmarking of three-qubit pure states on IBM's 127-qubit superconducting quantum processor, ibm_osaka. Focusing on the three-qubit W state, we exploit the theoretical result that almost all pure 3-qubit states are uniquely determined by any two of their two-qubit reduced density matrices. Using a 7-setting quantum state tomography protocol, we reconstruct 2-qubit marginal states so as to obtain the parent 3-qubit state. We further employ quantum steering ellipsoids to geometrically visualize the reconstructed subsystems. The experimentally obtained ellipsoid semi-axes lengths, centers and normalized volumes show good agreement with the theoretical predictions, demonstrating the usefulness of geometric benchmarking tools in the NISQ era.*

Keywords— *Quantum tomography; W state; Steering ellipsoids; IBM quantum processors*

1. Introduction

Quantum state reconstruction is essential for verifying and benchmarking quantum hardware. Although full tomography becomes impractical for large systems, theoretical insights allow global properties of a whole system to be inferred uniquely from its reduced system data [1,2,3]. It was shown [1,2,3] that almost all pure three-qubit quantum states are *uniquely* determined based on the information on *two* of their 2-qubit reduced density matrices.

On the other hand, *geometric picturization* provide an intuitive characterization of abstract quantum states. For example, the Bloch sphere provides a compact geometric representation [4] of a qubit. In this picture, pure quantum states of a qubit lie on the surface of the unit sphere, while mixed states occupy its interior, allowing a direct visualization of coherence and purity. Owing to its simplicity and intuitive appeal, the Bloch sphere has been extensively employed in the study of quantum dynamics and it plays a crucial role in quantum information processing. For 2-qubit systems, a deeper geometric insight arises through the concept of quantum steering ellipsoid [5]. It has been recognized that the entire collection of

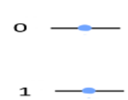
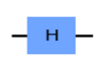
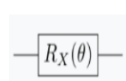
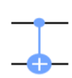
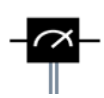
Bloch points associated with one of the qubits, which follow after a complete set of local projective measurements is performed on its partner qubit forms an ellipsoid, inscribed *within* the Bloch sphere. Signature features like volume, orientation, and displacement of the center of the ellipsoid representing the 2-qubit system reveal complete information about correlations, mixedness, and entanglement. This makes steering ellipsoids a powerful and experimentally accessible visualization tool for 2-qubit states.

In this paper we combine the geometric visualization in terms of steering ellipsoids associated with experimentally reconstructed 2-qubit states obtained using IBM's *ibm_osaka* quantum processor [6,7]. Reduced 2-qubit quantum state tomography is employed to reconstruct the parent 3-qubit W state, providing a proof-of-principle demonstration of whole pure 3-qubit state reconstruction from reduced 2-qubit subsystems, along with geometric benchmarking of the state using steering ellipsoids.

2. Quantum state tomography to reconstruct 2-qubit system

- We employ a set of 1-qubit gates i.e., Hadamard and X, Y, Z, rotation gate $R_X(\theta)$ and 2-qubit CNOT gate. The quantum circuit symbols and matrix representation of qubits, gates and measurements are illustrated in Table 1.

Table 1. Quantum circuit symbol and associated matrices for qubits, Hadamard, Rotation, CNOT gates and Z measurements.

Circuit symbol & Matrix Representation		
Qubit:		$\vert 0\rangle = \begin{pmatrix}1\\0\end{pmatrix}$ $\vert 1\rangle = \begin{pmatrix}0\\1\end{pmatrix}$
Hadamard Gate:		$H = \frac{1}{\sqrt{2}}\begin{bmatrix}1 & 1\\1 & -1\end{bmatrix}$
Rotation Gate:		$R_x(\theta)=\begin{pmatrix}\cos\frac{\theta}{2} & -i\sin\frac{\theta}{2}\\ -i\sin\frac{\theta}{2} & \cos\frac{\theta}{2}\end{pmatrix}$
CNOT Gate:		$\text{CNOT} = \begin{bmatrix}1&0&0&0\\0&1&0&0\\0&0&0&1\\0&0&1&0\end{bmatrix}$
Z measurements with outcomes 0, 1		$\Pi_0 = \begin{pmatrix}1&0\\0&0\end{pmatrix}$ $\Pi_1 = \begin{pmatrix}0&0\\0&1\end{pmatrix}$

- We employ a tomography scheme [7,8] consisting of a set of 7 quantum measurements to reconstruct an arbitrary 2-qubit system (see Table 2). A 2-qubit quantum state is characterized by a 4×4 real positive density matrix $\rho = \sum_{\alpha,\beta=1,2,3,4} \rho_{\alpha\beta} |\alpha\rangle\langle\beta|$ with Trρ=1 giving normalization condition. The probabilities of measurement outcomes given in column 2 of the Table 2 enable evaluation of elements of 2-qubit density matrix. Note that $I \otimes I$ corresponds to measurement in the computational (Z) basis of two qubits and it records 4 outcomes $\{i, j = 0, 1\}$ with a set of 4 probabilities $\vec{P}_{zz} = \{P_{zz}(i,j), i,j = 0,1\}$. This set of probabilities determine the diagonal elements of the density matrix i.e., $\vec{P}_{zz} \equiv (\rho_{11}, \rho_{22}, \rho_{33}, \rho_{44})^T$.
- All the off-diagonal elements of the density matrix are evaluated from the probabilities of outcomes of specific measurements given below:

$$P_{Hzz}(0,0) - P_{Hzz}(1,0) = 2\,\mathrm{Re}\rho_{13},\ P_{Hzz}(0,1) - P_{Hzz}(1,1) = 2\,\mathrm{Re}\rho_{24}$$
$$P_{zHz}(0,0) - P_{zHz}(1,0) = 2\,\mathrm{Re}\rho_{12},\quad P_{zHz}(0,1) - P_{zHz}(1,1) = 2\,\mathrm{Re}\rho_{34}$$
$$P_{zRz}(0,0) - P_{zRz}(1,0) = 2\,\mathrm{Im}\rho_{12}\quad P_{zRz}(0,1) - P_{zRz}(1,1) = 2\mathrm{Im}\rho_{34}$$
$$P_{Rzz}(0,0) - P_{Rzz}(1,0) = 2\,\mathrm{Im}\rho_{13}\quad P_{Rzz}(0,1) - P_{Rzz}(1,1) = 2\mathrm{Im}\rho_{24}$$
$$P_{CHzz}(0,0) - P_{CHzz}(1,0) = 2\,\mathrm{Re}\rho_{14}\quad (10)\quad P_{CHzz}(0,1) - P_{CHzz}(1,1) = 2\,\mathrm{Re}\rho_{23}$$
$$P_{CRzz}(0,0) - P_{CRzz}(1,0) = 2\,\mathrm{Im}\rho_{14}\qquad (12)\qquad P_{CRzz}(0,1) - P_{CRzz}(1,1) = 2\,\mathrm{Im}\rho_{23}.$$

- This 2-qubit tomography scheme is implemented experimentally on the 127-qubit superconducting quantum processor ibm_osaka accessed through the IBM quantum cloud platform [6]. Qubits q97, q98 and q99 of *ibm_osaka*, were selected due to their consistently low readout error and gate errors.
- We constructed the 3-qubit W state

$$|W_{ABC}\rangle = \frac{|1_A 0_B 0_C\rangle + |0_A 1_B 0_C\rangle + |0_A 0_B 1_C\rangle}{\sqrt{3}} \qquad (1)$$

  and carried out quantum state tomography (see Table. 2) to reconstruct *two* of the reduced 2-qubit density matrices ρ_{AB} and ρ_{BC} of the W state. Following the prescription given in references [2,7] for determining the whole pure 3-qubit state from two of its bipartite marginals we retrieve the W state. This provides a concrete demonstration of the theoretical result that any *two* of the 2-qubit marginals determine the full 3-qubit pure state uniquely.

Table 2. Quantum tomographic scheme for reconstructing 2-qubit state.

Tomographic operations	Experimental probabilities of z measurements on both the qubits with outcomes $\boldsymbol{i, j = 0, 1}$	Elements of the density matrix $\boldsymbol{\rho}$
$I \otimes I$	$\vec{P}_{zz} = \{P_{zz}(i,j) = \mathrm{Tr}(\rho\, \Pi_i \otimes \Pi_j)\}$	$\rho_{11}, \rho_{22}, \rho_{33}, \rho_{44}$
$H \otimes I$	$\vec{P}_{Hzz} = \{P_{Hzz}(i,j) = \mathrm{Tr}(\rho\, H\, \Pi_i\, H \otimes \Pi_j)\}$	$\mathrm{Re}\rho_{13}$, $\mathrm{Re}\rho_{24}$
$I \otimes R_X(\pi/2)$	$\vec{P}_{zRz} = \{P_{zRz}(i,j) = \mathrm{Tr}(\rho\, \Pi_i \otimes R_X(\pi/2)\, \Pi_j {R_X}^{\dagger}(\pi/2))\}$	$\mathrm{Im}\rho_{12}$, $\mathrm{Im}\rho_{34}$
$I \otimes H$	$\vec{P}_{zHz} = \{P_{zHz}(i,j) = \mathrm{Tr}(\rho\, \Pi_i \otimes H\, \Pi_j H\}$	$\mathrm{Re}\rho_{12}$, $Re\rho_{34}$
$R_X(\pi/2) \otimes I$	$\vec{P}_{Rzz} = \{P_{Rzz}(i,j) = \mathrm{Tr}(\rho\, R_X(\pi/2)\, \Pi_i {R_X}^{\dagger}(\pi/2)) \otimes \Pi_j\}$	$\mathrm{Im}\rho_{13}$, $\mathrm{Im}\rho_{24}$
$(H \otimes I)$ CNOT	$\vec{P}_{CHzz} = \{P_{CHzz}(i,j) = \mathrm{Tr}(\rho\, \mathrm{CNOT}\, H\, \Pi_i H \otimes \Pi_j\, \mathrm{CNOT}\}$	$\mathrm{Re}\rho_{14}$, $\mathrm{Re}\rho_{23}$
$(R_X(\pi/2)$	$\vec{P}_{CRzz} = \{P_{CHzz}(i,j)$	$\mathrm{Im}\rho_{14}$, $\mathrm{Im}\rho_{23}$

3. Geometric visualization in terms of steering ellipsoids

Our geometrical embedding is based on the 4×4 real matrix parametrization [5] Λ of the 2-qubit density matrix defined by $\rho = \frac{1}{4}\sum_{\mu,\nu=0}^{3} \Lambda_{\mu\nu}\, \sigma_\mu \otimes \sigma_\nu$, $\mu, \nu = 0, 1, 2, 3$. Here σ_0 denotes the 2×2 identity matrix and $\sigma_1, \sigma_2, \sigma_3$ are the standard Pauli matrices. The real parametrization

$$\Lambda = \begin{pmatrix} 1 & b^{\boldsymbol{T}} \\ a & T \end{pmatrix},$$

expressed in block form, consists of the Bloch vectors $(\boldsymbol{a}, \ \boldsymbol{b})$ of the constituent qubits and the correlation matrix $T = \left(t_{ij} = \mathrm{Tr}\,[\rho\, \sigma_i \otimes \sigma_j]\right), i, j = 1,2,3$.

Let us suppose that a 3-qubit state ρ_{ABC} is shared between three parties Alice, Bob and Charlie. We denote the reduced subsystems of Alice-Bob, Bob-Charlie, Alice-Charlie by ρ_{AB}, ρ_{BC}, ρ_{AC} respectively. Let Bob carry on a *set of all* projective measurements on his part of the qubit. It is shown [5] that the collection of collapsed Bloch vectors of Alice's qubit (obtained after Bob perform *all* possible projective measurements on their qubits) forms an ellipsoid, called the *steering ellipsoid* $\mathcal{E}_A$, centered at

$$c_A = \frac{a - Tb}{1 - b^2}$$

with orientation and semiaxes given by the eigenvectors, square roots of the eigenvalues (denoted by (s_1, s_2, s_3)) of the 3×3 real matrix

$$Q_A = \frac{1}{1-b^2}\left(T - ab^T\right)\left(I + \frac{bb^T}{1-b^2}\right)\left(T^T - ba^T\right).$$

Similarly, one obtains the ellipsoids $\mathcal{E}_B, \mathcal{E}_C$ for the collection of collapsed Bloch points of qubits at the end of Bob and Charlie. For the 3-qubit W state given in equation (1), the subsystem states are all identical; the steering ellipsoids are centered at $c_{\text{theoretical}} = (0,0,0.5)$ with semi-axes lengths given by $(s_1 = s_2 \approx 0.71, s_3 = 0.5)$. Normalized volume of the ellipsoid defined by $V = s_1 s_2 s_3$ of the subsystem steering ellipsoid is found to be $V_{\text{theoretical}} \approx 0.25$.

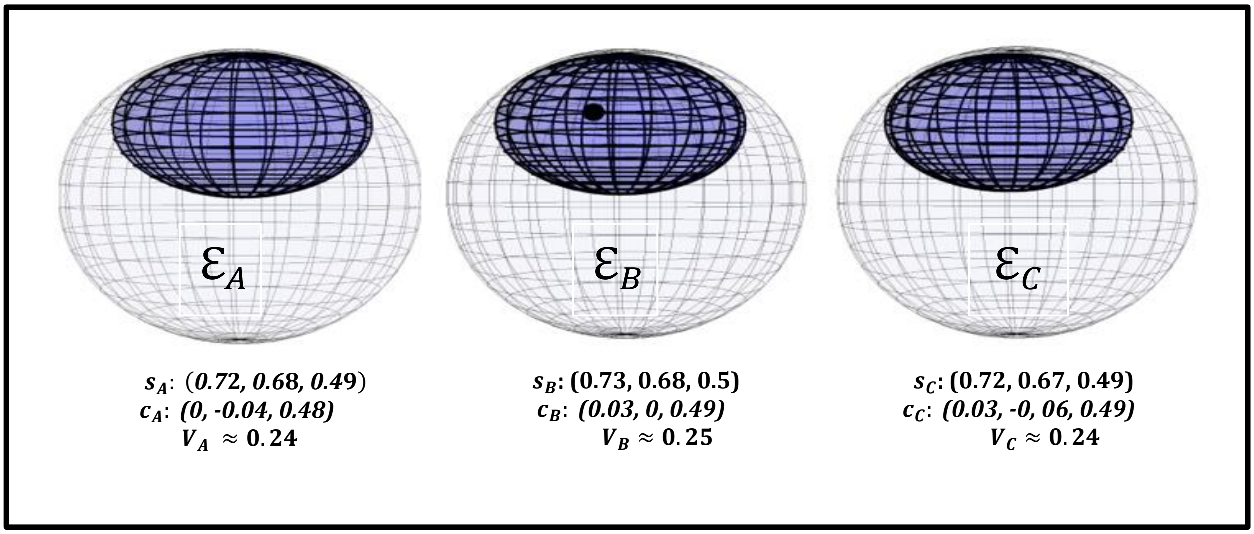


Figure 1: Steering ellipsoids, $\mathcal{E}_A, \mathcal{E}_B, \mathcal{E}_C$ the experimentally reconstructed 3-qubit W state. The ellipsoids are embedded within the Bloch sphere. The semi-axes lengths, centers and normalized volumes mentioned below each of the ellipsoids match with the theoretical counterparts within experimental limits.

We employ our experimental data to determine the center and semi-axes lengths of the steering ellipsoids of all the reduced states $\rho_{AB}, \rho_{BC}, \rho_{AC}$ enabling geometrical visualization of the 3-qubit W state. The set of 3 steering ellipsoids $\mathcal{E}_A, \mathcal{E}_B, \mathcal{E}_C$ associated with experimentally retrieved 3-qubit W state are depicted in Fig. 1. The semi-axes, normalized volumes and orientations of the steering ellipsoids closely match with the theoretical expectations for the W state, with small deviations attributed to noise and finite statistics. The consistency across different subsystems supports the reconstruction of the parent three-qubit state.

4. Conclusion

We have carried out quantum tomography of reduced 2-qubit systems of the 3-qubit W state using a set of 7 measurements on the qubits 97,98,99 of the 127 qubit IBM processor *ibmq_osaka.* The subsystem information enables recreation of the W state consistent with the theoretical result viz., *almost every pure three-qubit state can be determined completely by two of its two-qubit reduced density matrices*. The experimental data is employed to visualize the 3-qubit W state in terms of a set of three steering ellipsoids associated with the 2-qubit subsystems. Our study demonstrates the feasibility of mapping experimentally obtained quantum state data onto a geometric framework for pure 3-qubit states using current quantum hardware.

Acknowledgements

The authors acknowledge the use of IBM Quantum services for this work. The views expressed are those of the authors and do not necessarily reflect those of IBM Quantum. Sudha acknowledges support under the Q-Pragathi Project (No. QP202406) of the Quantum Research Park, Karnataka Innovation and Technology Society (KITS), K-Tech, Government of Karnataka

References

[1] N. Linden, S. Popescu, and W. K. Wootters, Phys. Rev. Lett. **89**, 207901 (2002).

[2] L. Díosi, Phys. Rev. A **70**, 010302 (2004).

[3] A. R. Usha Devi, A. K. Sudha, and S. Rajagopal, *Quan. Inf. Proc.* **11**, 685–710 (2011).

[4] M. A. Nielsen and I. L. Chuang, *Quantum Computation and Quantum Information* (10th Anniversary edition), Cambridge University Press, 2010.

[5] S. Jevtic, M. F. Pusey, D. Jennings, and T. Rudolph, Phys. Rev. Lett. 113, 020402 (2014).

[6] IBM Quantum, Ibm quantum experience (2021), https://quantum-computing.ibm.com

[7] Talath Humera, B. P. Govindaraja, B.G. Divyamani, Akshata Shenoy H, A. R. Usha Devi and Sudha, *Quantum Inf Process* **24**, 345 (2025).

[8] S. Dogra, A. A. Melnikov, G. S. Paraoanu, *Comm. Phys*. **4,** *26* (2021)